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UNIFORMLY COMPLEMENTED l_p^n's IN QUASI-REFLEXIVE BANACH SPACES

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ABSTRACT
It is shown that for every non-reflexive Banach space X with X^{**}/X reflexive there exists a uniformly bounded sequence of projections $\{P_n\}_{n=1}^{\infty}$ whose ranges are uniformly isomorphic to $\{l_p^n\}_{n=1}^{\infty}$ either for $p = 1$, or $p = 2$ or for $p = \infty$. The proof uses knowledge of the transfinite dual X^{ω}, ESA Schauder decompositions and proof of a similar statement for spaces with an unconditional basis due to Tzafriri.

It has been conjectured that each infinite dimensional Banach space X has uniformly complemented l_p^n's [8] (and see [12]). By X having uniformly complemented l_p^n's, we mean there is some constant K and some $p = 1$ or $p = 2$ or $p = \infty$ and a sequence $\{P_n\}_{n=1}^{\infty}$ of projections on X, so that for each n, $\|P_n\| \leqq K$ and $d(l_p^n, P_n(X)) \leqq K$. (Since l_2^n is uniformly complemented in $l_p^{2^n}$, for $1 < p < \infty$, the limitation that $p = 1$, 2 or ∞ is not of consequence.) Our main result is:

THEOREM 1. *If X is a non-reflexive Banach space, with X^{**}/X reflexive, then X has uniformly complemented l_p^n's.*

It is known that the infinite dimensional Banach space X has uniformly complemented l_p^n's if either

(i) X has an unconditional basis [14],

(ii) X has Lust and is not super-reflexive [6],

(iii) X is a subspace of a Banach lattice Y and l_∞ is not finitely representable in Y [7], or if

(iv) X is a special reflexive Banach space (see [11] for details).

Each of the above results requires some "unconditional-basis-like structure." In contrast, it is known that if X satisfies the hypothesis of Theorem 1, then X is not isomorphic to a subspace of any Y which has an unconditional basis, or more generally, any Banach lattice Y with property u.

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Theorem 1 also yields new partial positive answers to a question of Grothendieck [5] about the existence of non-nuclear operators (see Corollary 14).

The proof of Theorem 1 is given in Section 3. The proof makes use of properties of the transfinite dual X^{ω} of X and these results are in Section 1. In Section 2, we prove the following generalization of the result of Tzafriri's quoted in (i) above: subsymmetric Schauder decompositions have uniformly complemented l_p^n's.

§0. Preliminaries

We will adopt the Standard Definitions, Notations and Conventions of [10, p. xi–xiii] where all undefined terms may be found. We will write $\{x_n\}$, $[x_n]$, Σx_n for $\{x_n\}_{n=1}^{\infty}$, closed linear span $\{x_n\}_{n=1}^{\infty}$ and $\Sigma_1^{\infty} x_n$, respectively. All our spaces are normed and are usually complete.

Let X be a Banach space. A sequence $\{X_n\}$ at closed subspaces of X is called a *Schauder decomposition* [10, p. 47ff] if $X = [X_n]$ and there is a constant K, so that

$$\left\| \sum_1^p x_i \right\| \leqq K \left\| \sum_1^{p+q} x_i \right\|, \tag{0}$$

for all positive integers p, q and elements $x_i \in X_i$. The decomposition is said to be *unconditional* if there is a constant M so that

$$\left\| \sum_1^n b_i x_i \right\| \leqq M \left\| \sum_1^n x_i \right\| \tag{1}$$

for all positive integers n, elements $x_i \in X_i$, and scalars b_i with $|b_i| \leqq 1$. The smallest constant M which satisfies (1) is called the *unconditional constant* of the decomposition.

We will say that the decomposition $\{X_n\}$ of X is *fibered* by Y if there are isometries $\phi_n : Y \to X_n$ and $\theta_n : X_n \to Y$ with $\phi_n\theta_n = \theta_n\phi_n = \text{identity}$, for each n. A fibered decomposition $\{X_n\}$ of X is said to be *subsymmetric* if it is unconditional and if

$$\left\| \sum_1^n x_i \right\| = \left\| \sum_{i=1}^n \phi_{m(i)}(\theta_i x_i) \right\|, \tag{2}$$

for all positive integers n, increasing sequences of positive integers $\{m(i)\}$ and elements $x_i \in X_i$. Standard re-norming techniques will yield an equivalent subsymmetric norm on a subsymmetric decomposition whose unconditional constant is one (see [10, p. 114]). If $\{X_n\}$ is a subsymmetric decomposition, then the projection $T_n : X \to X$ which takes Σx_i onto $\Sigma_1^n \phi_i(w)$ has norm at most two, where $w = (\Sigma_1^n \theta_i x_i)/n$ (see [10, p. 116]).

We will say that the fibered decomposition $\{X_n\}$ of X is *ESA*, if for each n, m with $1 \leqq m \leqq n$ and elements $x_i \in X_i$, we have

$$\left\| \sum_1^n x_i \right\| \geqq \left\| \sum_1^{m-1} x_i + \sum_{m+1}^n x_i + w_m \right\| \tag{3}$$

when w_m is either $\phi_{m+1}(\theta_m x_m)$, $m \geqq 1$, or $\phi_{m-1}(\theta_m x_m)$, $m \geqq 2$. This definition implies the notion of ESA basis, as in [2], if the fiber is one-dimensional (see [2, p. 288] for a proof of this).

We also need the notion of the transfinite dual X^ω of a Banach space X as in [3]. Let X^0 be X and inductively define $X^{n+1} = (X^n)^*$, the dual of X^n. Let $J_n : X^n \to X^{n+2}$ be the canonical injection. We will identify X^n with its image under J_n in X^{n+2}. Then X^ω is defined to be the completion of the norm space $\bigcup X^{2n}$. We will use the following notation for adjoints, if $T^{(n)} : X^n \to Y^n$ then $T^{(n+1)} : Y^{n+1} \to X^{n+1}$ is the map $(T^{(n)})^*$, and $T^{(0)} = T$.

§1. Operators on X^ω

For any Banach spaces X and Y any bounded linear operator $T : X \to Y$, the diagram in Fig. 1 commutes (here J is the canonical injection) and $\| T^{(2)} \| = \| T \|$.

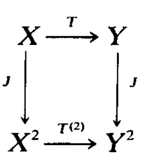

Fig. 1.

It follows now from Fig. 2, that there is a unique operator $T^\omega : X^\omega \to Y^\omega$ that extends each $T^{(2n)} : X^{2n} \to Y^{2n}$, furthermore $\| T^\omega \| = \| T \|$. (The downward pointing arrows are the canonical injections.)

In particular, for $T : X^{2n} \to X^{2m}$, between even duals of X, then T^ω maps X^ω into itself. Furthermore, if T is an isometry (respectively, projection, contraction) then T^ω is also an isometry (respectively, projection, contraction).

Actually it is not hard to see that Ω, which assigns X^ω to each Banach space X, and T^ω to each bounded operator T, is a covariant functor. (Note that $X^\omega \equiv X$, if X is reflexive.) To see this, let T, S, R be operators so that $T = SR$, then $T^{(2n)} = S^{(2n)} R^{(2n)}$ and hence $T^\omega = S^\omega R^\omega$. Furthermore, if $T : X \to X$ is id, the identity, then $T^\omega = \text{id}$ on X^ω.

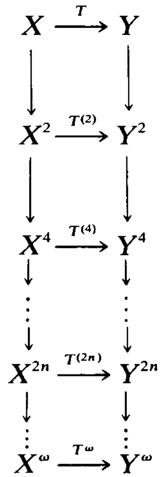

Fig. 2.

DEFINITION 2. Following Perrott [13], we define the isometries S_i for $i \geqq 0$ and the contractions Q_i for $i \geqq 1$ on X^ω by (see Fig. 3)

$$S_i = (J_{2i})^\omega, \quad \text{where } J_{2i} : X^{2i} \to X^{2i+2} \quad \text{and } i \geqq 0, \qquad \text{and} \tag{4}$$

$$Q_i = (J^{(1)}_{2i-1})^\omega, \quad \text{where } J^{(1)}_{2i-1} : X^{2i+2} \to X^{2i} \quad \text{and } i \geqq 1. \tag{5}$$

Our next result gives a "multiplication table" for these operators. It is well known that if X is non-reflexive, then the S_i's are not the id. In fact, $J_0^{(2)}$ and J_2 both: $X^2 \to X^4$ disagree on each $x \in X^2 \backslash J_0(X^0)$ was known by Dixmier [4] in 1948.

PROPOSITION 3.

$$S_j S_k = \begin{cases} S_{k+1} S_j & \text{if } j \leqq k, \\ S_k S_{j-1} & \text{if } j > k; \end{cases} \tag{6}$$

$$Q_j Q_k = \begin{cases} Q_{k-1} Q_j & \text{if } j < k, \\ Q_k Q_{j+1} & \text{if } j \geqq k; \end{cases} \tag{7}$$

$$S_j Q_k = \begin{cases} Q_{k+1} S_j & \text{if } j < k, \\ Q_k S_{j+1} & \text{if } j \geqq k; \end{cases} \tag{8}$$

$$Q_j S_k = \begin{cases} S_{k-1} Q_j & \text{if } j < k, \\ \text{id} & \text{if } j = k \text{ or } j = k+1, \\ S_k Q_{j-1} & \text{if } j > k+1. \end{cases} \tag{9}$$

$$
\begin{array}{ccc}
X^{2i} & \xrightarrow{J_{2i}} & X^{2i+2} \\
{\scriptstyle J_{2i}}\downarrow & & \downarrow{\scriptstyle J_{2i+2}} \\
X^{2i+2} & \xrightarrow{J^{(2)}_{2i}} & X^{2i+4} \\
{\scriptstyle J_{2i+2}}\downarrow & & \downarrow{\scriptstyle J_{2i+4}} \\
X^{2i+4} & \xrightarrow{J^{(4)}_{2i}} & X^{2i+6} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
X^{2i+2k} & \xrightarrow{J^{(2k)}_{2i}} & X^{2i+2k+2} \\
{\scriptstyle J_{2i+2k}}\downarrow & & \downarrow{\scriptstyle J_{2i+2k+2}} \\
X^{2i+2k+2} & \xrightarrow{J^{(2k+2)}_{2i}} & X^{2i+2k+4} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
X^{\omega} & \xrightarrow{S_i} & X^{\omega}
\end{array}
\qquad
\begin{array}{ccc}
X^{2i+2} & \xrightarrow{J^{(1)}_{2i-1}} & X^{2i} \\
{\scriptstyle J_{2i+2}}\downarrow & & \downarrow{\scriptstyle J_{2i}} \\
X^{2i+4} & \xrightarrow{J^{(3)}_{2i-1}} & X^{2i+2} \\
{\scriptstyle J_{2i+4}}\downarrow & & \downarrow{\scriptstyle J_{2i+2}} \\
X^{2i+6} & \xrightarrow{J^{(5)}_{2i-1}} & X^{2i+4} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
X^{2i+2k+2} & \xrightarrow{J^{(2k+1)}_{2i-1}} & X^{2i+2k} \\
{\scriptstyle J_{2i+2k+2}}\downarrow & & \downarrow{\scriptstyle J_{2i+2k}} \\
X^{2i+2k+4} & \xrightarrow{J^{(2k+3)}_{2i-1}} & X^{2i+2k+2} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
X^{\omega} & \xrightarrow{Q_i} & X^{\omega}
\end{array}
$$

Fig. 3.

PROOF. These relations are obtained by looking at the definition of S_i and Q_i (see Fig. 3), their transposes and using the functor Ω. The exception is the middle line of (9), which we do in detail.

It is well known and easy to check that the composite map:

(10) $$X^{2i} \xrightarrow{J_{2i}} X^{2i+2} \xrightarrow{J^{(1)}_{2i-1}} X^{2i}$$

is the identity. Hence $Q_iS_i = (J^{(1)}_{2i-1})^{\omega}(J_{2i})^{\omega} = \mathrm{id}$.

Each statement like (10) is a dual statement obtained by subtracting one from everything and taking transposes. For (10) we have

$$X^{2i-1} \xrightarrow{J_{2i-1}} X^{2i+1} \xrightarrow{J^{(1)}_{2i-2}} X^{2i-1}$$

is the id (here i must be greater than one). Transposing we obtain

$$X^{2i} \xleftarrow{J^{(1)}_{2i-1}} X^{2i+2} \xleftarrow{J^{(2)}_{2i-2}} X^{2i}$$

whose composition is also the id. Hence $\mathrm{id} = (J^{(1)}_{2i-1})^\omega (J^{(2)}_{2i-2})^\omega = Q_i S_{i-1}$, for $i \geqq 1$, since $(J^{(2)}_{2i-2})^\omega = (J_{2i-2})^\omega = S_{i-1}$. This completes the middle line of (9).

To obtain (6), look at the kth box in the definition of S_i in Fig. 3. We have

$$J_{2i+2k+2} J^{(2k)}_{2i} = J^{(2k+2)}_{2i} J_{2i+2k}$$

which by Ω yields $S_{i+k+1} S_i = S_i S_{i+k}$, $i, k \geqq 0$. Inspection will show this is equivalent to (6).

The dual to the k-th above yields $Q_i Q_{i+k+1} = Q_{i+k} Q_i$, $i \geqq 1$, $k \geqq 0$ which is equivalent to (7).

Now looking at the k-th box in the definition of Q_i in Fig. 3 we obtain

$$S_{i+k} Q_i = Q_i S_{i+k+1}, \qquad i \geqq 1, \quad k \geqq 0$$

which yields half of (8) and (9). The other half follows from the dual of this k-th box

$$S_i Q_{i+k+1} = Q_{i+k+2} S_i, \qquad i \geqq 0, \quad k \geqq 0.$$

This completes the proof.

Corollary 4.

(11) $$S_1^i Q_1^i S_1^j Q_1^j = S_1^n Q_1^n, \qquad n = \max\{i, j\}, \quad \text{if } i, j \geqq 0;$$

(12) $$S_{n+k} S_1^n Q_1^n = S_1^n S_k Q_1^n = S_1^n Q_1^n S_{n+k}, \qquad \text{if } n \geqq 0, \quad k \geqq 1;$$

(13) $$Q_{n+k+1} S_1^n Q_1^n = S_1^n Q_{k+1} Q^n = S_1^n Q_1^n Q_{n+k+1}, \qquad \text{if } n \geqq 0, \quad k \geqq 1;$$

(14) $$Q_{k+1} S_1^k Q_1^k = Q_1 S_1^k Q_1^k \quad \text{and} \quad S_{k+1} S_1^k Q_1^k = S_1 S_1^k Q_1^k, \qquad \text{if } k \geqq 0;$$

(15) $$S_1^k Q_1^k Q_{k+1} = S_1^k Q_1^{k+1}, \qquad \text{if } k \geqq 0.$$

(16) *Let $n > j \geqq 1$. Then $TV(S_1^{j-1} Q_1^{j-1} - S_1^j Q_1^j) = 0$ when $V = S_n$ or Q_n and $T = S_1^j Q_1^j$ or $I - S_1^{j-1} Q_1^{j-1}$.*

Proof. The equations (11)–(15) are straightforward applications of Proposition 3. To see (16), consider the special case, $V = Q_n$, $T = S_1^{n-1} Q_1^{n-1}$. Now (16) follows from (15) and $S_1^{n-1} Q_1^n S_1^{n-2} Q_1^{n-2} = S_1^{n-1} Q_1^n = S_1^{n-1} Q_1^n S_1^{n-1} Q_1^{n-1}$. Otherwise, (12) or (13) implies T commutes with V and $V(S_1^{j-1} Q_1^{j-1} - S_1^j Q_1^j) = 0$ follows from (11).

Remark. One can obtain similar relations for general $T: X^{2n} \to X^{2m}$ from the definition of T^ω. Namely

$$T^\omega S_{n+k} = S_{m+k} T^\omega \quad \text{and} \quad Q_{m+k+1} T^\omega = T^\omega Q_{n+k+1}, \qquad k \geqq 0.$$

The next Lemma is most likely known, a proof is included for completeness.

LEMMA 5. *If* $S, Q : X \to X$ *are norm one operators, with* S *an isometry and* $QS = \mathrm{id}$, *then the kernel of the norm one projection* S^nQ^n *is the direct sum* $Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n$, *where* $Y_1 = \ker Q_1$ *and* $Y_{i+1} = S(Y_i)$.

PROOF. Let $P_n = S^nQ^n$. Since $S^nQ^nS^nQ^n = S^n \,\mathrm{id}\, Q^n = P_n$, we have that P_n is a projection, similarly $P_nP_m = P_nP_m = P_{\max(m,n)}$. Thus the kernel of P_n is the direct sum

$$(I - P_1)(X) \oplus (P_1 - P_2)(X) \oplus \cdots \oplus (P_{n-1} - P_n)(X).$$

Since S is an isometry, $\ker Q = \ker P_1 = Y_1$. We complete, the proof by showing $Y_n = (P_{n-1} - P_n)(X)$ by induction. If $y \in Y_n$, there is $x \in Y_1$ so that $S^{n-1}x = y$. Since $P_{n-1}S^{n-1}x = S^{n-1}(\mathrm{id})x = y$ and $P_nS^{n-1}x = S^nQ\,\mathrm{id}\,x = S^n0 = 0$, we have one inclusion. Suppose $x = P_{n-1}x - P_nx = S(P_{n-2} - P_{n-1})Qx$. Taking Q of both sides implies $Qx \in (P_{n-2} - P_{n-1})(X) = Y_{n-1}$ and so $x = P_1x \in Y_n$ since $(P_{n-1} - P_n)P_1 = P_{n-1} - P_n$.

COROLLARY 6. *Let* $Y = Y_1 =$ *kernel of* Q_1 *in* X^ω *and let* $Y_{i+1} = S_1(Y_i)$. *For* $y \in Y$, *define* $y_1 = y$ *and* $y_{i+1} = S_1(y_i)$.

Then $S_1^i(y_j) = y_{i+j}$, $Q_1(y_1) = 0$, *and* $Q_1(y_{i+1}) = y_i$, *for* $i \geqq 1$.

Furthermore, $S_i(y_j) = y_{j+1}$, $Q_i(y_j) = y_{j-1}$, *if* $j \geqq i > 1$ *and* $S_i(Y_j), Q_i(Y_j) \subset Y_j$, *if* $1 \leqq j < i$.

PROOF. The Furthermore statement follows from (14) and (16) of Corollary 4.

Unfortunately, $S_k \mid Y_j$ need not be the identity for $1 \leqq j < k$.

PROPOSITION 7. *The following are equivalent:*

(i) $S_2 \mid Y_1$ *is the identity.*

(ii) $T \mid Y_j$ *is the identity, for* $T = S_n$ *or* Q_n *and* $n > j \geqq 1$.

(iii) $S_n \mid Y_1$ *is the identity, for some* $n \geqq 1$.

(iv) *The space* X^2/X^0 *is reflexive.*

PROOF. Since $S_{n+1}S_n = S_nS_n$, we have $S_n \mid Y = \mathrm{id}$ implies $S_{n+1} \mid Y = \mathrm{id}$. Conversely, if $S_{n+1} \mid Y = \mathrm{id}$ and $S_n(Y) \subset Y$, it follows that S_n is a projection on Y. Since S_n is an isometry, $\ker S_n = \{0\}$ and hence $S_n \mid Y = \mathrm{id}$. Therefore, (iii) implies (i) implies $S_n \mid Y_1 = \mathrm{id}$ for $n > 1$.

This implies (ii) for $T = S_n$, since by (12), we have $S_n \mid Y_j = S_1^{j-1}(S_{n-j+1} \mid Y_1)Q_1^{j-1}$. Since $Q_nS_n = \mathrm{id}$, (ii) is true for $T = Q_n$ as well. Therefore (i), (ii) and (iii) are equivalent.

Now consider $Z = X^2/X^0$ and let $T: X^2 \to Z$ be the quotient map. It is well known [4], that $T^{(2)}: X^4 \to Z^2$ has a right inverse $V: Z^2 \to X^4$ so that V is an isometry and the short exact sequence

$$0 \longrightarrow Z^2 \xrightarrow{\;V\;} X^4 \xrightarrow{\;J_1^{(1)}\;} X^2 \longrightarrow 0$$

splits via $J_2: X^2 \to X^4$. It follows that $V^{(2)}(Z^4) = \ker J_j^{(3)}$, $V^{(2)}(Z^4) \cap J_4(X^4) = V(Z^2)$ and that the short exact sequence

$$0 \longrightarrow Z^\omega \xrightarrow{\;V^\omega\;} X^\omega \xrightarrow{\;Q_1\;} X^\omega \longrightarrow 0$$

splits via $S_1: X^\omega \to X^\omega$.

Now if Z is not reflexive, then $Z^4 \neq Z^2$ and there is $x \in \ker J_1^{(3)} \backslash J_4(X^4) \subset Y_1$. Hence $S_2 x = J_4^{(2)} x \neq J_6 x = x$ and so (i) is false. Thus (i) implies (iv). Conversely, if Z is reflexive, $Y_i = \ker Q_1 = V^\omega(Z^\omega) = V(Z^2)$ and S_2 is the identity on all of X^4 and the proof is complete.

REMARK. We note that S_2 defined for X^ω restricted to $V^\omega(Z^\omega)$ yields S_1, defined for Z^ω, since

$$\begin{array}{ccc} Z^2 & \xrightarrow{\;V\;} & X^4 \\ {\scriptstyle J_2}\big\downarrow & & \big\downarrow{\scriptstyle J_4} \\ Z^4 & \xrightarrow{\;V^{(2)}\;} & X^6 \end{array}$$

commutes.

PROPOSITION 8. *If X is non-reflexive but X^2/X^0 is reflexive then using the notation of Corollary 6, the fibered decomposition $\{Y_i\}$ is ESA.*

PROOF. If $\{y^i\}_{i=1}^n \subset Y = Y_1$, let $y_j^i = S_1^{j-1}(y^i)$. We must show that the norm of $z = \sum_{i=1}^n y_1^i$ is at least as big as the norm of each of the following elements:

$$\sum_{1}^{m-1} y_i^i + \sum_{M+1}^{n} y_i^i + w_m, \tag{17}$$

for $w_m = y_{m+1}^m$ or y_{m-1}^m, $1 \leqq m \leqq n$ [set $y_0^1 = 0$].

This is easy using the results of Corollary 6 and Proposition 7 since either $Q_{m+2} S_m z$ or $S_m Q_m z$ is the vector (17) and these operators have norm one.

REMARK. If X^2/X^0 is one-dimensional, this is a result of Perrott [13]. In fact, the proof of Proposition 8 is very much like that in [13].

Proposition 9. *A fibered* ESA *decomposition has a complemented subspace with a fibered subsymmetric decomposition.*

Proof. For the reader's convenience we will prove this using the notations used in the proof of Proposition 8. Note that the operators S_i and Q_i, $i \geqq 1$ are well-defined on the span $\{Y_i\}$. First we show these operators must have norm one.

To see $\|S_i\| \leqq 1$, note

$$\left\| \sum_{j=1}^{n} y_j^j \right\| \geqq \left\| \sum_{j=1}^{n-1} y_j^j + y_{n+1}^n \right\| \geqq \cdots \left\| \sum_{j=1}^{i-1} y_j^j + \sum_{j=i}^{n} y_{j+1}^j \right\| = \left\| S_i \sum_{j=1}^{n} y_j^j \right\|.$$

Actually S_i is an isometry since these steps are reversible. The norm of Q_i is one since

$$\left\| \sum_{j=1}^{n} y_j^j \right\| = \left\| \sum_{j=1}^{i-1} y_j^j + y_{i-1}^i + \sum_{j=i+1}^{n} y_j^j \right\| = \left\| \sum_{j=1}^{i-1} y_j^j + y_{i-1}^i + \sum_{j=i+1}^{n} y_{j-1}^j \right\|$$

which is $\|Q_i(\Sigma_{j=1}^n y_j^j)\|$.

Let $Z_i \subset Y_{2i-1} + Y_{2i}$ be defined by $Z_i = \{y_{2i-1} - y_{2i} : y \in Y\}$. Note that $\{Z_i\}$ is a block decomposition of the $\{Y_i\}$, fibered by Z_1. We claim that $[Z_i]$ is the desired subspace.

First $\{Z_i\}$ is orthogonal, hence unconditional, for reasons similar to the proof of this result for ESA basis in [2]. To illustrate this, we show how to remove the middle term of $A = \|z_1^1 - z_2^1 + z_3^2 - z_4^2 + z_5^3 - z_6^3\|$. This norm is equal to the norm of each of

$$z_1^1 - z_2^1 + z_i^2 - z_{i+1}^2 + z_{N+4}^3 - z_{N+5}^3, \qquad 3 \leqq i \leqq N+2.$$

Averaging we have

$$\begin{aligned} A &\geqq \|z_1^1 - z_2^1 + (z_3^2 - z_{N+3}^2)/N + z_{N+4}^3 - z_{N+5}^3\| \\ &= \|z_1^1 - z_2^1 + (z_3^2 - z_4^2)/N + z_5^3 - z_6^3\| \to \|z_1^1 - z_2^1 + z_5^3 - z_6^3\|. \end{aligned}$$

The decomposition is subsymmetric, which can be seen by applying suitable repetitions of the isometries S_i's as in [13].

Finally the projection P is given by

$$P\left(\sum_{i=1}^{\infty} y_i^i\right) = \frac{1}{2} \sum_{i=1}^{\infty} (y_{2i-1}^{2i-1} - y_{2i-1}^{2i} + y_{2i}^{2i} - y_{2i}^{2i-1}).$$

It is straightforward to check that $P^2 = P$. To see that P is bounded, it suffices to show $I - P$ is bounded. Now

$$(I-P)\left(\sum_{i=1}^{\infty} y_i^i\right)=\frac{1}{2}\sum_{i=1}^{\infty}(y_{2i-1}^{2i-1}+y_{2i-1}^{2i}+y_{2i}^{2i}+y_{2i}^{2i-1})$$

$$=\frac{1}{2}[(\cdots S_6Q_6S_4Q_4S_2Q_2)+(\cdots S_5Q_6S_3Q_4S_1Q_2)]\left(\sum_{i=1}^{\infty} y_i^i\right),$$

which clearly has norm one. This completes the proof.

REMARK. The operator $\frac{1}{2}[(\cdots S_6Q_6S_4Q_4S_2Q_2)+(\cdots S_5Q_6S_3Q_4S_1Q_2)]$ is a well-defined norm one projection in each X^ω. To see this note that the collection $\{S_{2n}Q_{2n}, S_{2n-1}Q_{2n} : n = 1,2,\cdots\}$ is a collection of commuting projections except for $S_{2n}Q_{2n}S_{2n-1}Q_{2n} = S_{2n}Q_{2n}$ and $S_{2n-1}Q_{2n}S_{2n}Q_{2n} = S_{2n-1}Q_{2n}$. Since $S_{2n}Q_{2n}$ is the identity on X^{2n}, the infinite composition $\cdots S_4Q_4S_2Q_2$ is a well-defined norm one operator on $\bigcup X^{2n}$ which is dense in X^ω. If P, R are the two infinite compositions above we have $PR = P$ and $RP = R$ and the result follows.

§2. Complemented l_p^n's in subsymmetric decompositions

In [14], Tzafriri proved that Banach spaces with an unconditional basis have uniformly complemented l_p^n's for either $p = 1,2$ or $p = \infty$. Proposition 11 shows how to modify Tzafriri's proof to handle the case for Banach spaces with a subsymmetric decomposition. There are three points in which the proof below differs from that of Tzafriri. We have no need for Ramsey's Theorem, proposition 5 of [14] needs the addition of Lemma 10 and case III of theorem 1 of [14] needs a different argument.

LEMMA 10. *If $\{y_i\}_1^N \subset X$ and $\{y_i^*\}_1^N \subset X^*$ are a bi-orthogonal sequence. Suppose there is a constant K so that for any scalars $\{a_i\}$,*

$$\left\|\sum_1^N a_iy_i\right\| \leqq K\left(\sum_1^N |a_i|^2\right)^{1/2}$$

and

$$\left\|\sum_1^N a_iy_i^*\right\| \leqq K\left(\sum_1^N |a_i|^2\right)^{1/2}.$$

then the projection $Px = \sum_1^N y_i^*(x)y_i$ has norm $\leqq K^2$ and $d(P(X), l_2^n) \leqq K^2$.

PROOF. Straightforward.

PROPOSITION 11. *If X has a subsymmetric decomposition $\{Y_n\}$ with fiber Y, then X has uniformly complemented l_p^n's.*

PROOF. We follow the proof of theorem 1 of [14]. As in [14] we may assume that the unconditional constant is one, and we identify three cases.

Let $y \in Y$ (respectively $y^* \in Y^*$) be a norm one element, define $\lambda(y, n) = \|\Sigma_1^n y_i\|$ (respectively $\mu(y^*, n) = \|\Sigma_1^n y_i^*\|$).

Case I. For each $h > 1$ there are $y \in Y$ with $\|y\| = 1$ and n so that $\lambda(y, hn)/\lambda(y, n) < 2$. Then as in [14], there is are l_∞^n's uniformly complemented in X.

Case II. For each $h > 1$ there are $y^* \in Y^*$ with $\|y^*\| = 1$ and n so that $\mu(y^*, hn)/\mu(y^*, n) < 2$. Then similar to case II in [14], there are uniformly complemented l_1^n's in X.

Case III. All others. Using proposition 4 of [14], we have the existence of a constant A and $q > 2$ so that for any norm one elements $y \in Y$ or $y^* \in Y^*$ we have

$$\left\|\sum_1^n a_i y_i\right\| \Big/ \lambda(y, n) \leqq A\left(\sum_1^n |a_i|^q\right)^{1/q} \Big/ n^{1/q}, \qquad \text{and} \tag{18}$$

$$\left\|\sum_1^n a_i y_i^*\right\| \Big/ \mu(y^*, n) \leqq A\left(\sum_1^n |a_i|^q\right)^{1/q} \Big/ n^{1/q} \tag{19}$$

for any scalars $\{a_i\}$. [Here A, q are independent of y, y^* and n.]

To complete the proof it suffices to find y^*, y with $y^*(y) = 1$ and lower Lq'-estimates for (18) and (19), where $1/q' + 1/q = 1$. This follows since the Rademacher elements of $[y_i]_1^{2^n}$ and $[y_i^*]_1^{2^n}$ satisfy the hypothesis of Lemma 10 (see proposition 5 of [14]).

To do this let n be given and let $N = 2^n$ and define $B = \inf\{\lambda(y, N)\colon y \in Y, \|y\| = 1\}$. Choose $y \in Y$, so that $\|y\| = 1$ and $B \leqq \lambda(y, N) \leqq 2B$. Let $y^* \in Y^*$, $\|y^*\| = y^*(y) = 1$, clearly $\mu(y^*, N)\lambda(y, N) \geqq N$. Let $z \in X$ with $\|z\| = 1$ and $(\Sigma_1^N y_i^*)(z) \geqq \mu(y^*, N)/2$. By the 1-unconditionality of $\{Y_n\}$, we may assume there are $\{z^i\}_1^N \subset Y$ so that $z = \Sigma_1^N \phi_i(z^i)$. From section 0, $w = \Sigma_1^N z^i/N$ satisfies $\|\Sigma_1^N w_i\| \leqq 2$ and $(\Sigma_1^N y_i^*)(\Sigma_1^N w_i) = (\Sigma_1^N y_i^*)(z)$.

We need to show $\|w\|$ is small. Let $v = w/\|w\|$. Now $\lambda(v, N) = \|\Sigma_1^N v_i\| = \|\Sigma_1^N w_i\|/\|w\|$. Thus $\|w\|\lambda(v, N) \leqq 2$. So $\|w\| \leqq 2/B \leqq 4/\lambda(y, N)$ and $\mu(y^*, N) \leqq 2(\Sigma_1^N y_i^*)(z) = 2Ny^*(w) \leqq 2N\|w\| \leqq 8N/\lambda(y, N)$. It is now straightforward to check that for any scalars $\{a_i\}$, $\|\Sigma_1^N a_i y_i\|/\lambda(y, N) \geqq (\Sigma_1^N |a_i|^{q'})/8AN^{1/q'}$ and $\|\Sigma_1^N a_i y_i^*\|/\mu(y, N) \geqq (\Sigma_1^N |a_i|^{q'})/8AN^{1/q'}$. This completes the proof.

§3. The theorem and applications

PROOF OF THEOREM 1. Let X be non-reflexive with X^2/X^0 reflexive. By the principle of local reflexivity [8, p. 196] and by denseness of $\bigcup X^{2n}$ in X^ω, finite rank projections on X^ω can be pulled down to X. Hence it suffices to prove the theorem for X^ω.

Using the notation of Corollary 6, by Lemma 5 $[Y]_1^{2N}$ is 2-complemented in X^ω. Further, by Proposition 9, $[Z_i]_1^N$ is 2-complemented in $[Y_i]_1^{2N}$. Thus an appeal to Proposition 11 completes the proof.

For a Banach space X define $R(X) = X^2/X^0$, and define $R_1(X) = R(X)$, $R_{i+1}(X) = R(R_i(X))$ (as in [3]).

COROLLARY 12. *If X is non-reflexive but for some integer k, $R_k(X)$ is reflexive (or equivalently $R_{k+1}(X) = \{0\}$), then X has uniformly complemented l_p^n's.*

PROOF. The theorem applies to $R_{k-1}(X)$ hence its bidual $(R_{k-1}(X))^2$ has uniformly complemented l_p^n's. Since $(R(X))^2$ is complemented in X^4, local reflexivity completes the proof.

COROLLARY 13. *If X is non-reflexive, but l_1 is not finitely representable in X, then X has uniformly complemented l_2^n's.*

PROOF. In [3], it is shown that if the hypothesis of Corollary 12 is false, then l_1 is finitely representable in X. Hence X has uniformly complemented l_p^n's. Now p must be 2, since l_1 is finitely representable in the sequence $\{l_1^n\}$ or $\{l_\infty^n\}$ (see [10, p. 97]).

COROLLARY 14. *If either X or Y satisfies the hypothesis of the theorem, then there are compact non-nuclear maps from X to Y and Y to X.*

PROOF. See proposition IV.4 of [6].

CLOSING REMARKS. Consider the general case of X non-reflexive. There are two ways to try to generalize the above proof to this case.

(i) Just define $\bar{S}_i$ (respectively, $\bar{Q}_i$) to be the identity on $[Y_j]_{j=1}^{i-1}$ and to be S_1 (respectively, Q_1) on $[Y_j]_{j=i}^{\infty}$. It is easy to see that these new operators have norm $\leqq 3$. The proof of Proposition 8 requires that these operators have norm one. However, the theorem holds if X can be renormed so that the $\bar{S}_i$'s and $\bar{Q}_i$'s have norm one on X^ω.

(ii) In Proposition 7, we observed that S_2 restricted to the kernel of Q_1 (in X^4) $= X^{1\perp}$ (in X^4) is the identity. Thus if we let $W_1 = X^{1\perp}$ (in X^ω) (or more precisely $\cdots J_8 J_6 J_4(X^{1\perp})$) and let $W_{i+1} = S_1(W_i)$, then $\{W_i\}$ is a fibered ESA decomposition. Each W_i is complemented in Y_i, however this yields an unworkable estimate of the norm of the projection of $[Y_i]_{i=1}^N$ onto $[W_i]_{i=1}^N$. If $[W_i]_{i=1}^N$ are uniformly complemented in X^ω then the theorem holds.

The author does not know if either method will work in general.

Added in proof. Additional progress on the uniformly complemented l_p^n-conjecture has been made by G. Pisier (*Holomorphic semi-groups and the geometry of Banach spaces*, preprint). Pisier proves that the conclusion of Corollary 13 is still true without the hypothesis that X is non-reflexive. In particular, super-reflexive Banach spaces have uniformly complemented l_2^n's. This result together with standard ultraproduct constructions or nonstandard analysis imply that if the uniformly complemented l_p^n-conjecture is false, then there is a non-reflexive Banach space X without uniformly complemented l_p^n-spaces. Such a space X must be "infinitely" non-reflexive in the sense that X violates the hypothesis of Corollary 12.

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